Figure 3 gives the fields of flow past a combination of two spheres and a cylinder, at $M_\infty = 0.8$, for various values of the aspect ratio. Evolution of the flow is well illustrated. On Fig. 3a the body is nearly spherical and a single supersonic zone appears in its vicinity. On increasing the length of the cylinder (Fig. 3b) the supersonic zone splits into two distinct zones, the second of which is situated downstream and contains a stronger shock than the first one, although the shock is still weaker than that appearing in Fig. 3a. Figure 3c depicts the case when the cylindrical part of the body is still longer. Here two weak supersonic zones appear which are spaced even further apart.

Figure 4 shows a flow past a combination of two spheres and a 10% cone, again at $M_\infty = 0.8$. Here the supersonic zone is situated at the rear part of the body. The flow first accelerates on the front sphere reaching $M \approx 0.8$, then slows down to $M \approx 0.66$ and flows past the cone with very slowly increasing velocity.

Computations are also performed for a flow past a 10% spherically truncated cone with various ellipsoidal tailpieces. The distribution of parameters along the body up to some small distance from the point of attachment of the ellipsoid are practically identical with those of the case shown on Fig. 4.

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NONSYMMETRIC MECHANICS OF TURBULENT FLOWS

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In [1] we proposed renouncing the hypothesis of a symmetric tensor of Reynolds stresses and an agitated fluid and introducing an equation of conservation of the moment of momentum. This equation turns out to be nontrivial if, for example, the pulsed momentum transfer through a flow cross section depends on the orientation of the cross section in space.

In the present paper we derive the equations of nonsymmetrical mechanics of turbulent flows by converting from integral conservation laws (under the assumption that the micromotions are described by the Navier-Stokes equation). The phenomenological characteristics of the agitated fluid are introduced naturally as the average values (for the fluid mass element in question) of the corresponding microcharacteristics or their fluxes. The closing kinetic hypotheses on the internal kinetic moment (of the turbulent vortices) and pulsed momentum and moment-of-momentum transfer are formulated.

1. The laws of conservation of mass, momentum, and moment of momentum for the Euler volume V bounded by the surface S are of the form [2]

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho u_n dS = 0 \tag{1.1}$$

$$\int_V \frac{\partial (\rho u_i)}{\partial t} dV + \int_S \rho u_i u_n dS = \int_V \rho F_i dV + \int_S t_{in} dS \tag{1.2}$$

$$\int_V \frac{\partial}{\partial t} (\varepsilon_{ijk} \rho u_j x_k)\, dV + \int_S \varepsilon_{ijk} \rho u_j x_k u_n dS = \int_V (\rho G_i + \rho \varepsilon_{ijk} x_j F_k)\, dV + \int_S (c_{in} + \varepsilon_{ijk} x_j t_{kn})\, dS \tag{1.3}$$

Here ρ is the density of the fluid particle, u_i is its velocity, F_i is the body force, t_{ij} is the stress tensor, $\mathbf{x}$ (x_k) is the radius vector of the fluid particle, G_i is the body moment, c_{ij} are the moment stresses, and ε_{ijk} is the Levi-Civita alternating tensor. The subscript n identifies components lying along the normal to the surface element dS.

The interpretation of the dynamic quantities occurring in the integrands of (1.1)–(1.3) depends on the chosen scales of the associated microvolumes dV, i.e. on the size of the fluid particles for which the quantities ρ, u_i, etc. are determined. If the motion of these particles is described by the Navier-Stokes equations and if the microvolumes dV are sufficiently small, then

$$t_{ij} \equiv t_{ji} = \left(-p + \frac{2}{3} \rho \nu \frac{\partial u_k}{\partial x_k}\right) \delta_{ij} + \rho \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.4}$$

$$G_i = 0, \qquad c_{ij} = 0$$

For simplicity we shall disregard the action of the body forces, assuming that $F_i \equiv 0$.

If we base our analysis on the consideration of several large fluid particles, then the balance of moments of momenta reduces to the vorticity diffusion equation, and the quantities c_{ij}, for example, are determined by molecular vorticity transfer (see [1]).

Let the volume V be filled with an agitated fluid, i.e. with fluid containing turbulent vortices which comprise a special (*) microstructure of a scale larger than the differential volume $dV = dx_1 dx_2 dx_3$. We assume that the linear scale of the turbulent vortices is somewhat smaller than the characteristic length Δ of the volume V, and that the density $\langle \rho \rangle$ and mass velocity U_i averaged over the volume V, namely

*) In contrast to the microstructure of solids, the microstructure under consideration here varies randomly both in space and in time.

$$\langle\rho\rangle = \frac{1}{V}\int_V \rho dV, \qquad \langle\rho\rangle U_i = \langle\rho u_i\rangle = \frac{1}{V}\int_V \rho u_i dV \tag{1.5}$$

as well as the average quantities defined below, are no longer random parameters.

As usual, we refer the velocity U_i to the center of mass of the fluid filling the volume V, i.e. to the point with the radius vector $\mathbf{X}$ (X_1, X_2, X_3) defined as follows:

$$\langle\rho\rangle X_i = \frac{1}{V}\int_V \rho x_i dV, \qquad \int_V \rho\xi_i dV = 0 \tag{1.6}$$

Here $\xi_i = x_i - X_i$ is the vector which defines the position of a point inside the volume V relative to the center of mass of the latter.

Let the field of average velocities U_i (X_j, t) introduced above be such that the characteristic linear scale of the velocity gradient is of the same order as L. From now on we shall confine our attention to volumes V whose linear dimensions Δ are much smaller than L.

In the present paper we deal with motions of an agitated fluid such that the microparticle velocity u_i (ξ_j, t) can be expressed as the sum of a regular and a nonregular component,

$$u_i(\xi_j, t) = \left(U_i + \frac{\partial U_i}{\partial X_j}\xi_j + O\left(\frac{\Delta^2}{L^2}\right)\right) + v_i(\xi_j, t) \tag{1.7}$$

where v_i is the turbulent pulsation (the nonregular component) of the velocity, and $L \gg \Delta \gg \xi_j \gg 0$.

By (1.5)–(1.7) we have

$$\int_V \rho v_i dV = 0$$

Further assumptions concerning the field $v_i(\xi_j, t)$ are formulated below.

2. Now let us suppose that our volume V is the macrovolume element

$$V = \Delta X_1 \Delta X_2 \Delta X_3.$$

Equations (1.1)–(1.3) then become:

$$\frac{\partial\langle\rho\rangle}{\partial t} + \frac{\partial}{\partial X_j}\langle\rho u_j\rangle_j = 0 \tag{2.1}$$

$$\frac{\partial\langle\rho u_i\rangle}{\partial t} + \frac{\partial}{\partial X_j}\langle\rho u_i u_j\rangle_j = \frac{\partial\langle t_{ij}\rangle_j}{\partial X_j} \tag{2.2}$$

$$\frac{\partial}{\partial t}\langle\varepsilon_{ijk}\rho u_j x_k\rangle + \frac{\partial}{\partial X_j}\langle\varepsilon_{ilk}\rho u_l x_k u_j\rangle_j = \frac{\partial}{\partial X_j}\langle\varepsilon_{ilk} t_{lj} x_k\rangle_j \tag{2.3}$$

Here $\langle\varphi_{ij}\rangle_j$ denotes the result of averaging φ_{ij} over an area whose normal is the axis X_j, i.e. over one of the faces of the volume V.

We have already used volume integration of equations valid for micromotions and introduced quantities averaged over volumes and surfaces for the analysis of the dynamics of heterogeneous media (for example, see our paper [3]). Eringen [4] investigated this approach in general and compared it with the micropolar elasticity and anisotropic fluid models.

Taking the vector product of Eq. (2.2) and $\mathbf{X}$, we obtain the balance of the moment of translational momentum,

$$\frac{\partial}{\partial t}(\varepsilon_{ijk}\langle\rho u_j\rangle X_k)+\frac{\partial}{\partial X_j}(\varepsilon_{ilk}\langle\rho u_l u_j\rangle_j X_k)-$$
$$-\varepsilon_{ilk}\langle\rho u_l u_k\rangle_k=\frac{\partial}{\partial X_j}(\varepsilon_{ilk}\langle t_{lj}\rangle_j X_k)-\varepsilon_{ilk}\langle t_{lk}\rangle_k \qquad (2.4)$$

Subtracting relation (2.4) from Eq. (2.3), we obtain the equation of conservation of the internal moment of momentum for the volume element,

$$\frac{\partial}{\partial t}\langle\varepsilon_{ijk}\rho u_j\xi_k\rangle+\frac{\partial}{\partial X_j}\langle\varepsilon_{ilk}\rho u\,\xi_k u_j\rangle_j+$$
$$+\varepsilon_{ilk}\langle\rho u_l u_k\rangle_k=\frac{\partial}{\partial X_j}\langle\varepsilon_{ilk}t_{lj}\xi_k\rangle_j+\varepsilon_{ilk}\langle t_{lk}\rangle_k \qquad (2.5)$$

Let us replace the average characteristics and their fluxes in Eqs. (2.1),(2.2),(2.5) by the phenomenological parameters of turbulent flow, i.e. by velocity U_i, the Reynolds stress tensor R_{ji}, the viscous stress tensor T_{ij}, the internal moment of inertia J_{ij}, the internal rotational velocity of the turbulent vortices ω_j, and the moment stress tensor μ_{ij}.

We begin with the assumption that the results of averaging the vector quantities over the volume V and over its faces are equivalent, i.e. that

$$\langle\rho u_i\rangle_j=\langle\rho u_i\rangle=\langle\rho\rangle U_i,\qquad \langle\varepsilon_{ilk}\rho u_l\xi_k\rangle_j=\langle\varepsilon_{ilk}\rho u_l\xi_k\rangle \qquad (2.6)$$

etc. This implies, among other things, the possibility of the transformation

$$\langle\rho u_i u_j\rangle_j=\langle\rho\rangle U_i U_j-R_{ij}+o\left(\frac{\Delta}{L}\right) \qquad (2.7)$$

which introduces the Reynolds stresses

$$R_{ij}=-\langle\rho v_i\;v_j\rangle_j \qquad (2.8)$$

into momentum equation (2.2).

It is important to note that the tensor R_{ij} is generally nonsymmetric. The problem of the symmetry (or nonsymmetry) of the tensor R_{ij} can be solved (as is usual in the mechanics of contiuous media) by analyzing the moment-of-momentum equation. We note that Reynolds distinguished between the components R_{ij} and R_{ji}. in his original paper [5].

If we average the pulsation velocities v_i and v_j over time (as is done in measuring the microstructure of turbulent flows), the resultant tensor $v_i v_j$ is the correlation moment of the pulsed velocity field; in general it does not coincide with R_{ij}.

Statistical averaging (over the ensemble of possible realizations)of the Navier-Stokes equations yields balance relations for the average fluid motion in a volume element of the scale dx_i (but not dX_i). We note that the elements of the problem of statistical and volume averaging are presented in [6]. The form of the ergodic hypothesis (of the equivalence of all averaging methods) which is usually employed excludes the investigation of nonsymmetric effects.

Let us consider the average kinetic moment of an agitated fluid. We have

$$\langle\varepsilon_{ilk}\rho u_l\xi_k\rangle=\frac{1}{V}\int_V\varepsilon_{ilk}\rho u_l\xi_k dV=$$
$$=\varepsilon_{ilk}\frac{\partial U_l}{\partial X_m}\frac{1}{V}\int_V\rho\xi_m\xi_k dV+\frac{1}{V}\int_V\varepsilon_{ijk}\rho v_j\xi_k dV \qquad (2.9)$$

Further transformation require us to make additional assumptions about the pulsed field v_j. We assume that the volume under consideration can be broken down into a set of small volumes ΔV with the characteristic scale $2d$ such that in each of them we have

$$v_j(\xi_m, t) = w_j + \frac{\partial w_j}{\partial \bar{\xi}_m} \zeta_m \tag{2.10}$$

where $d \geqslant \zeta_m \geqslant 0$, $\bar{\xi}_m = \xi_m - \zeta_m$ are the coordinates of the center of mass of the small volume ΔV and $w_j = v_j(\bar{\xi}_m)$ is the velocity of this center of mass, i. e.

$$\frac{1}{\Delta V}\int_{\Delta V} \rho v_j dV = \bar{\rho} w_j, \qquad \frac{1}{\Delta V}\int_{\Delta V} \rho \xi_m dV = \bar{\rho}\bar{\xi}_m$$

$$\int_{\Delta V} \rho \zeta_m dV = 0, \qquad \frac{1}{\Delta V}\int_{\Delta V} \rho dV = \bar{\rho}$$

This enables us to express integrals (2. 9) as sums over the volumes ΔV ,

$$\langle \varepsilon_{ilk} \rho u_l \xi_k \rangle = \varepsilon_{ilk} \frac{\partial U_l}{\partial X_m} \frac{1}{V} \sum \bar{\rho} \bar{\xi}_m \bar{\xi}_k \Delta V +$$

$$+ \frac{1}{V} \sum \varepsilon_{ilk} w_l (\bar{\xi}_m, t)\, \bar{\xi}_k \bar{\rho} \Delta V + \frac{1}{V} \sum \varepsilon_{ilk} \left(\frac{\partial U_l}{\partial X_m} + \frac{\partial w_l}{\partial \bar{\xi}_m} \right) \int_{\Delta V} \rho \zeta_m \zeta_k dV \tag{2.11}$$

We shall limit our attention to the case where $\Delta \gg d$, where the sums in (2.11) can be replaced by integrals, and where the radius vectors $\bar{\xi}_m$ vary continuously $(dV = d\bar{\xi}_1 d\bar{\xi}_2 d\bar{\xi}_3)$

$$\langle \varepsilon_{ilk} \rho u_l \xi_k \rangle = \varepsilon_{ilk} \frac{\partial U_l}{\partial X_m} \left(\frac{1}{V} \int_V \bar{\rho} \bar{\xi}_m \bar{\xi}_k dV \right) +$$

$$+ \frac{1}{V} \int_V \varepsilon_{ilk} w_l (\bar{\xi}_m, t)\, \bar{\xi}_k \bar{\rho} V + M_i \tag{2.12}$$

The average internal moment M_i in this expression is a volume moment, i. e.

$$M_i = \frac{1}{V} \int_V \varepsilon_{ilk} \Phi_{lm} i_{mk} dV = \langle \varepsilon_{ilk} \Phi_{lm} i_{mk} \rangle$$

$$\Phi_{lm} = \left(\frac{\partial U_l}{\partial X_m} + \frac{\partial w_l}{\partial \bar{\xi}_m} \right), \qquad i_{mk} = \frac{1}{\Delta V} \int_{\Delta V} \rho \zeta_m \zeta_k dV = O(d^2) \tag{2.13}$$

and i_{mk} is the moment of inertia of a fluid particle in the volume ΔV.

In contrast to (2. 9), expression (2 12) contains the average kinetic moment of small-scale turbulent vortices in isolated form. We assume that the contribution of the pulsed velocity field is confined to the volume moment M_i, i. e. to

$$\frac{1}{V} \int_V \varepsilon_{ilk} w_l (\bar{\xi}_m, t)\, \bar{\xi}_k \bar{\rho} dV = 0$$

If it is necessary to isolate the contributions made by vortices of various scales to the kinetic moment of the volume V, we can continue the procedure of conversion from (2. 9) to (2. 12) by assuming that the centers of the small vortices also experience rotations of some scale $d_* \ (\Delta \gg d_* \gg d)$.

We note that the isolation of volumes d in a continuous velocity field is somewhat arbitrary; however, a change in scale entails changes in the magnitudes of the associated velocity field gradients, so that the kinetic moment M_i remains unchanged. In [1] the corresponding moment of inertia was set equal to the moment of inertia of the fluid volume V ; this procedure did not reflect the volume character of the moment M_i.

The first term of (2.12) can be developed into

$$\varepsilon_{ilk}\frac{\partial U_l}{\partial X_m}\left(\frac{1}{V}\int\limits_V \bar{\rho}\,\bar{\xi}_m\bar{\xi}_k dV\right) = \varepsilon_{ilk}\,\frac{\partial U_l}{\partial X_m} I_{mk}$$

where $I_{mk} = I_{km}$ is the specific moment of inertia of the fluid in the volume element V. This term is of the order of $U\Delta^2 / L$ (where U is the magnitude of the average velocity). We assume that the fluid is sufficiently agitated to make M_i quite large (even for $\Delta \gg d$) and thus to allow us to neglect the first term. Then, finally,

$$\langle \varepsilon_{ilk}\rho u_l \xi_k\rangle \approx \langle \varepsilon_{ilk}\Phi_{lm} i_{mk}\rangle = M_i \tag{2.14}$$

where $i_{mk} = i_{km}$ is a symmetric tensor. If the volumes ΔV of the particles in pulsed rotation are symmetric, then $i_{mk} = {}^1/_2\, i\delta_{mk}$ and we have

$$M_k = \langle i\Phi_k\rangle \tag{2.15}$$

where $\Phi_k = \Omega_k + \Phi_k^*$ is the total angular velocity of the internal rotations. Since $\langle\Phi\rangle = \Omega = {}^1/_2 \operatorname{rot} U$ by virtue of (1.7), it follows that Φ^* is its pulsed component. We can introduce the average values of the specific moment of inertia J and of the effective internal rotation velocity ω_i of the turbulent vortices,

$$M_i = \langle (J + i^*)\,(\Omega_i + \Phi_i^*)\rangle = J\,(\Omega_i + \omega_i) \tag{2.16}$$

$$J = \langle i\rangle, \quad \omega_i = \langle i^*\Phi_i^*\rangle\, J^{-1}$$

Here i^* is the pulsation of the specific moment of inertia, and we can introduce the pulsation ω_i^* of the proper angular velocity of the vortex,

$$J\omega_i^* = i^*\Phi_i^* - J\omega_i$$

Further, principle (2.6) enables us to transform the kinetic moment fluxes into

$$\langle \varepsilon_{ilk}\rho u_l \xi_k u_j\rangle_j = M_i U_j - \mu_{ij} + o\left(\frac{\Delta}{L}\right) \tag{2.17}$$

which introduces the moment stresses

$$\mu_{ij} = -\,\langle i\Phi_i v_j\rangle_j \tag{2.18}$$

into moment-of-momentum balance equation (2.5).

With the same degree of accuracy we have

$$\varepsilon_{ilk}\,\langle \rho u_l u_k\rangle_k = \varepsilon_{ilk}\,\langle \rho v_l v_k\rangle_k = -\,\varepsilon_{ilk}\,R_{lk} \tag{2.19}$$

Averaging of the viscous stresses t_{kp} in the incompressible case reduces to the transformation

$$\frac{1}{V}\int\limits_S t_{kn} dS = \frac{\partial\langle t_{kp}\rangle_p}{\partial X_p} = \frac{\partial}{\partial X_p}\left\langle -p\delta_{kp} + \nu\rho\left(\frac{\partial u_k}{\partial \xi_p} + \frac{\partial u_p}{\partial \xi_p}\right)\right\rangle_p =$$

$$= \frac{\partial}{\partial X_p}\left\{-P\delta_{kp} + \nu\rho\left(\frac{\partial U_k}{\partial X_p} + \frac{\partial U_p}{\partial X_k}\right) + \nu\rho\left\langle\left(\frac{\partial v_k}{\partial \xi_p} + \frac{\partial v_p}{\partial \xi_k}\right)\right\rangle_p\right\}, \qquad P = \langle p\rangle_p$$

by virtue of (1.4), (1.7). Neglecting the effect of the pulsed velocity field on the average viscous stresses, we obtain

$$T_{kp} = \langle t_{kp}\rangle_p = -\,P\delta_{kp} + \nu\rho\left(\frac{\partial U_k}{\partial X_p} + \frac{\partial U_p}{\partial X_k}\right) \tag{2.20}$$

We note that the presence of pulsed velocities can affect the appearance of the antisymmetric component $\varepsilon_{ilk}\,T_{lk}$ (see (2.5)). However, we shall not consider this effect here.

We shall also neglect the moment of the viscous stresses $\langle \varepsilon_{ilk} t_{lj} \xi_k \rangle_j$ acting at the faces of the volume V.

3. The final system of equations of nonsymmetric mechanics of turbulent flow of an incompressible fluid is of the form

$$\frac{\partial U_j}{\partial X_j} = 0, \qquad \rho\left(\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial X_j}\right) = -\frac{\partial P}{\partial X_i} + \frac{\partial R_{ij}}{\partial X_j} \tag{3.1}$$

$$\frac{\partial M_i}{\partial t} + \frac{\partial M_i U_j}{\partial X_j} = \frac{\partial \mu_{ij}}{\partial X_j} + \varepsilon_{ilk} R_{lk}$$

The dynamic variables which appear in the balance equations of the momentum and internal moment of momentum must be related to the kinematic variables U_i, ω_i by certain equations which are specific to nonsymmetric hydromechanics in the case under consideration [7]. Following Boussinesq, we assume that the scalar coefficients in the defining relations are functions of the average microstructural parameters of the agitated fluid. The defining relations in this case are

$$\frac{1}{2}(R_{ij} + R_{ji}) = \varepsilon\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$$

$$\frac{1}{2}(R_{ij} - R_{ji}) = -2\gamma\varepsilon_{ijk}\omega_k, \quad \gamma > 0$$

$$\mu_{ij} = (2\alpha\delta_{ij}\delta_{km} + 2\beta\delta_{im}\delta_{jk} + 2\eta\delta_{ik}\delta_{jm})\frac{\partial(\Omega_k + \omega_k)}{\partial X_m} \tag{3.2}$$

Here δ_{ij} is a unit tensor, ε is the coefficient of turbulent shear viscosity, γ is the coefficient of turbulent rotational viscosity, and α, β, η are the coefficients of turbulent gradientially vortical viscosity (our terminology differs from that of [7]). Determination of the transfer coefficients ε, γ, α, β, η requires us to introduce hypotheses on mixing kinetics in turbulent flow. One way to do this is to use the ideas of Taylor and Prandtl [8, 9] on the existence of a characteristic displacement length analogous to the free path length in the kinetic theory of gases.

We shall carry out the appropriate analysis for a plane free turbulent flow, namely for the steady plane flow characterized by the conditions

$$U_1 = U_1(X_1, X_2),\ U_2 = U_2(X_1, X_2),\ U_3 = 0$$

$$\Omega_1 = \Omega_2 = 0, \ \ \Omega_3 = {}^1/_2\,(\partial U_2/\partial X_1 - \partial U_1 / \partial X_2) = \Omega \tag{3.3}$$

In addition we shall assume that the velocity pulses have an average orientation such that

$$\omega_1 = \omega_2 = 0, \qquad \omega_3 = \omega(X_1, X_2) \tag{3.4}$$

In his monograph [10] Schlichting notes that "vortices with axes parallel to the direction of flow predominate in flows along a wall; vortices with axes perpendicular to the direction of principal flow and to the direction of the velocity gradient predominate in free turbulence". In view of this we assume that conditions (3.3) together with (3.4) correspond to freely turbulent flows.

In flows along a plane wall we have $\omega_1 \neq 0$, $\omega_2 = \omega_3 = 0$, and the equations of momentum and moment of momentum become separable.

System (3.1), (3.2) then assumes the form

$$\frac{\partial U_1}{\partial X_1}+\frac{\partial U_2}{\partial X_2}=0$$

$$\rho\left(U_1\frac{\partial}{\partial X_1}+U_2\frac{\partial}{\partial X_2}\right)U_i=-\frac{\partial P}{\partial X_i}+\frac{\partial R_{i1}}{\partial X_1}+\frac{\partial R_{i2}}{\partial X_2}$$

$$\left(U_1\frac{\partial}{\partial X_1}+U_2\frac{\partial}{\partial X_2}\right)[J(\Omega+\omega)]=\frac{\partial\mu_{31}}{\partial X_1}+\frac{\partial\mu_{32}}{\partial X_2}+R_{12}-R_{21} \qquad (3.5)$$

$$R_{12}+R_{21}=2\varepsilon\left(\frac{\partial U_1}{\partial X_2}+\frac{\partial U_2}{\partial X_1}\right),\qquad R_{11}=-R_{22}=\varepsilon\frac{\partial U_1}{\partial X_1}$$

$$R_{12}-R_{21}=-4\gamma\omega$$

$$\mu_{11}=\mu_{22}=\mu_{33}=\mu_{12}=\mu_{21}=0,\quad \mu_{31}=2\eta\frac{\partial(\Omega+\omega)}{\partial X_1},\quad \mu_{32}=2\eta\frac{\partial(\Omega+\omega)}{\partial X_2}$$

We shall also consider flows for which the velocity components can be expressed as

$$U_1=U_\infty-U,\quad U_2=W$$
$$U_\infty/U\gg1,\qquad W/U\approx L_2/L_1\ll1$$

where $U_\infty=\mathrm{const}$ and L_1, L_2 are the flow scales along the axes X_1, X_2. The estimates associated with the boundary layer approximations [10] then enable us to simplify system (3,5) considerably,

$$-U_\infty\frac{\partial U}{\partial X_1}=\frac{\partial}{\partial X_2}\left(-\varepsilon_0\frac{\partial U}{\partial X_2}-2\gamma_0\omega\right)$$
$$U_\infty\frac{\partial}{\partial X_1}J(\Omega+\omega)=-4\gamma_0\omega+\frac{\partial}{\partial X_2}2\eta_0\frac{\partial}{\partial X_2}(\Omega+\omega) \qquad (3.6)$$

where $\varepsilon_0=\varepsilon\rho^{-1}$, $\gamma_0=\gamma\rho^{-1}$, $\eta_0=\eta\rho^{-1}$ are the corresponding "kinematic" turbulent viscosities and $\Omega={}^1/_2\,\partial U/\partial X_2$.

4. Now let us formulate the hypotheses concerning the transfer coefficients ε, γ, η and the moment of inertia J in a turbulent flow. To this end let us consider the expressions for the Reynolds stress components and for the moment stress in terms of the pulsations,

$$r_{12}=-\rho\langle v_1v_2\rangle_2,\quad \mu_{32}=-\rho\langle i\Phi v_2\rangle_2,\quad J=\langle i\rangle \qquad (4.1)$$

since $\Phi_1=\Phi_2=0$, $\Phi=\Phi_3$ in the case under consideration.

We shall estimate quantities (4.1) on the basis of some idealized picture of motion of an "average" fluid microelement in turbulent flow (i.e. by computing its average pulsations). The translational velocity pulses v_1, v_2 will be estimated from the difference between the average velocities U in neighboring flow layers lying the small distance l away from each other. Such an estimate [8, 9] is due to the possibility of isolating the average displacement path l during whose traversal the momentum of a fluid microelement is conserved [9]. In traveling the distance between the indicated layers the migrating microelement generates velocity pulses because its velocity differs from the velocity of the aborigine particles. Since $U\gg W$, the estimates are carried out with respect to the component U and under the assumption of an isotropic distribution of the absolute pulsations of the translational velocity $|v_1|\sim|v_2|$.

We also assume in this case that the momentum transfer due to the difference between the average translational velocities in the layers $X_2 = \text{const}$, $X_2 + l = \text{const}$ separated by the "free path" length l also yields equal estimates of the magnitudes of the pulsations v_1 and v_2. Thus, fluid microelements pass through the boundary X_2 =const of the layer, and the aforementioned difference Δ_U between the momenta of the arriving and departing particles is given by

$$\Delta_U(\rho v_1) = \rho U_1(X_2 + l) - \rho U_1(X_2) = \rho l \frac{\partial U_1}{\partial X_2} \tag{4.2}$$

$$|\Delta_U v_2| \sim |\Delta_U v_1| \sim l \left| \frac{\partial U_1}{\partial X_2} \right|, \quad \rho = \text{const}, \quad l > 0 \tag{4.3}$$

However, the "average" micromotion is now associated with the existence of an "average" field of internal angular velocities ω. We assume accordingly that a migrant particle passes through a vortex sheet of intensity $l_* \omega$ in traversing the path l. The particles which pass the sheet in opposite directions then carry an additional momentum proportional to the sheet intensity,

$$\Delta_\omega(\rho v_1) = \rho l_* \omega - (-\rho l_* \omega) = 2\rho l_* \omega \tag{4.4}$$

It is important to note that the presence of the vortex sheet affects only the tangential velocity components (the normal components remain unaltered). Hence,

$$\rho v_1 = \rho l \frac{\partial U_1}{\partial X_2} + 2\rho l_* \omega, \qquad |v_2| = l \left| \frac{\partial U_1}{\partial X_2} \right| \tag{4.5}$$

Making use of estimates (4.5) and allowing for the choice [10] of the sign of v_2 (the stress must be of the same sign as the transferred quantity), we infer from (4.1) that

$$r_{12} = \rho l \left| \frac{\partial U_1}{\partial X_2} \right| \left(l \frac{\partial U_1}{\partial X_2} + 2 l_* \omega \right) \tag{4.6}$$

Comparison with formulas (3.5) yields an estimate for the turbulent shear and vortical viscosities,

$$\varepsilon = \rho l^2 \left| \frac{\partial U_1}{\partial X_2} \right|, \qquad \gamma = \rho l l_* \left| \frac{\partial U_1}{\partial X_2} \right| \tag{4.7}$$

We note that the estimate of the component $r_{21} = -\rho \langle v_2 v_1 \rangle_1$ differs from the derivation of the expression for r_{12} in the following way. We consider the layer $X_1 = \text{const}$. The particle migrating through this layer is characterized by an additional increment (equal to $2 l_* \omega$) in the tangential velocity component (namely v_2). It is important to recognize, however, that the sign of the increment is negative (the traversal of the field by the migrant particle is now in a direction orthogonal to the previous one), i.e. we have

$$|v_1| = l \left| \frac{\partial U_1}{\partial X_2} \right|, \qquad \rho v_2 = \rho l \frac{\partial U_1}{\partial X_2} - 2\rho l_* \omega \tag{4.8}$$

$$r_{21} = \rho l \left| \frac{\partial U_1}{\partial X_2} \right| \left(l \frac{\partial U_1}{\partial X_2} - 2 l_* \omega \right) \tag{4.9}$$

Now let us estimate the moment of momentum transferred into the layer $X_2 = \text{const}$ by a migrant particle "freely" traversing a path of length l.

We can now compute the pulsation of the transferred moment of momentum,

$$(i\Phi)^* = i\Phi - \langle i\Phi \rangle = (i^* + J)(\Phi^* + \Omega) - J(\Omega + \omega) =$$
$$= i^*\Phi^* - J\omega + J\Phi^* = J(\omega^* + \Phi^*) \tag{4.10}$$

From this we can estimate the kinetic moment $(i\Phi)^*$ carried by pulsed particle transfer through the surface (of thickness l) bounding the layer $X_2 = \text{const}$,

$$\omega^* = \omega(X_2 + l) - \omega(X_2) = l\frac{\partial\omega}{\partial X_2}, \qquad \Phi^* = \Omega(X_2 + l) - \Omega(X_2) = l\frac{\partial\Omega}{\partial X_2}$$
$$(i\Phi)^* = Jl\frac{\partial}{\partial X_2}(\omega + \Omega) \tag{4.11}$$

Applying the same criteria as above in choosing signs, we obtain the following expression for the moment stress:

$$\mu_{32} = Jl^2\left|\frac{\partial U_1}{\partial X_2}\right|\frac{\partial}{\partial X_2}(\omega + \Omega) \tag{4.12}$$

This gives us the following expression for the gradientally vortical viscosity:

$$\eta = \frac{1}{2}Jl^2\left|\frac{\partial U}{\partial X_2}\right| \tag{4.13}$$

We note that the considerations used in deriving estimate (4.13) can be traced back to Taylor's idea [8, 11] on pulsed vorticity transfer.

To compute the specific moment of inertia J of the turbulent vortices we assume that the volume V of the agitated fluid under consideration is filled with fluid particles of "average" radius d. The average specific moment of inertia J (equal to the ratio of the polar moment of inertia i on the "average" particle to its volume in the general case and to its area in the particular case of plane flow). We then have the estimate

$$J = {}^1/_2\, d^2 \tag{4.14}$$

We see therefore that the microstructure of an agitated fluid is characterized by three parameters: the mixing length l, the diameter $2d$ of a rotating microparticle, and the width l_* of the vortex sheet. It is apparently justifiable to assume that only one parameter of the "state" of the turbulent microstructure (e.g. $l_* = Al$, $d = Bl$, where A and B are numerical coefficients) can be independent.

Estimation of the numerical coefficients A and B requires either experimental data or hypothetical refinements of the picture of average micromotion (e.g. the assumption that the mixing length is equal to the microparticle diameter, $l = 2d$, so that $A = 1$).

5. The theory of turbulent flows contains an average equation for the vorticity [11] obtained by applying the $^1/_2$ rot and averaging operations (assuming that all averaging operations are equivalent) successively to the Navier-Stokes equations. For example, Townsend [12] writes out this equation for steady three-dimensional flow and notes that it has the same form as the conservation equation in the plane case where (for example) only the components $\Omega_3 = \Omega$, $\Phi_3 = \Phi$ differ from zero and where all the variables are functions of X_1, X_2 ,

$$\frac{\partial}{\partial X_j}(\Omega U_j) = -\frac{\partial}{\partial X_j}\overline{\Phi v_j} + \nu\nabla^2\Omega, \qquad j = 1, 2 \tag{5.1}$$

The Taylor vorticity transfer equation (5.1) can also be obtained by applying the procedure of averaging over the volume V to the equation of vorticity diffusion Φ in the plane case where the latter is of divergent form. This requires us to make the substitution $\overline{\Phi v_j} = \langle \Phi v_j \rangle_j$ in Eq. (5.1) Thus, construction of the vorticity balance for a volume V of agitated fluid does not reveal the existence of an average angular velocity of proper rotation of the turbulent vortices ω (a kinematically independent quantity). Simple averaging of the vorticity equation and its weighted averaging (i.e. averaging of the kinetic moment) yield differing results. In these cases where $\omega = 0$ either the vortex balance equations (5.1) or the equation for the moment of momentum in system (2.5) must follow from the momentum balance equation.

It is important to note that in constructing semiempirical theories of turbulence Taylor [11] and later researchers regarded vorticity transfer equation (5.1) as a substitute for the momentum equation (while noting the need to introduce an independent kinetic hypothesis). Only Mattioli [13, 14] seems to have adopted a more general view; he approached the average momentum and moment-of-momentum equations as fundamental independent turbulent flow equations. Limiting himself to the hydraulic formulation for the analysis of turbulent flow in a circular pipe, Mattioli introduced additional characteristics of a turbulent fluid, namely the vorticity, the moment of inertia, and the moment of internal forces Γ. However, he assumed that the vorticity is kinematically related to the average velocity field, that the moment of inertia is a constant, and that the moment of internal forces Γ is proportional to the derivative of the vorticity. He then used the above condition of independence of the equations to eliminate the turbulent viscosity (to determine the displacement length l).

The novel elements in Mattioli's study did not receive the attention they deserved. Von Karmán [15], while taking note of Mattioli's "interesting theory of turbulent transfer", qualified his praise by adding that the Mattioli forces were "not readily comprehensible"; Mattioli's papers are not even entered in the bibliography appended to the encyclopedic monograph on the theory of turbulence by authors of [16].

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